Antenna Boresight Parameter Estimation

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A boresight-calibration test is devised based upon the functional dependence of signal-strength measurements upon the antenna pointing relative to the direction to a ground-based transmitter. Algorithms are herein developed for the estimation of the antenna-boresight orientation relative to the spacecraft-attitude reference axes, the antenna beamwidths, and the boresight signal strength. Expressions for the expected error in the boresight parameter estimation are stated. Results of a computer simulation of the estimation process are included; they agree with the theoretically expected performance. Formulas are included for the transformation of attitude and orbital parameters to determine the antenna-pointing direction.

Nomenclature

a = semi-major axis of spacecraft orbit
$\bar{\mathbf{b}}$ = unit vector along the antenna boresight
e,i = orbital eccentricity, inclination
K = Earth gravitational constant
$\bar{l}$ = unit vector from spacecraft to ground-based transmitter
l_Ω = subsatellite longitude at τ_Ω
m_i,t_i = measurement and time of measurement
S = signal strength expressed in decibels (db)
S_o = co-boresight signal strength
t = time
(u,v,w) = nominal antenna-fixed reference axes with w along nominal antenna boresight
(x,y,z) = spacecraft attitude reference axes
(X_I,Y_I,Z_I) = Earth centered, inertial coordinates with Z_I along the ascending node and Y_I along the polar axis
(X_o,Y_o,Z_o) = orbital coordinates with Z_o along the spacecraft nadir direction and Y_o opposite the orbit normal
$\bar{\alpha},\hat{\alpha}$ = antenna parameter vector, best estimate of the antenna parameter vector
γ = constant signal strength contour rotation angle about boresight axis
δ_u,δ_v = antenna half-power beam widths
λ_T,l_T = ground transmitter latitude and longitude
ν = true anomaly
τ_π = time of perigee passage
τ_Ω = time of crossing the ascending node
ψ,Φ,θ = Euler angles with sequence yaw, roll, and pitch about spacecraft Z, X, and Y axes, respectively
ω = argument of perigee
ω_E = sidereal angular velocity

Introduction

LARGE, directional, orbital-based communication antennas often require the on-orbit evaluation of boresight parameters. This evaluation is especially desired if the antenna possesses any of the following properties: 1) narrow beam width, 2) on-orbit deployment, 3) sensitivity to deformation, or 4) open-loop pointing control. This paper reports a technique for the estimation of antenna boresight parameters, namely, the two beam-pointing parameters with respect to the spacecraft attitude (sensor) reference axes, the beam widths, and the antenna gain. Emphasis is placed upon a mathematical model leading up to computer processing of redundant data.

Received January 15, 1971; revision received June 7, 1971.
Index categories: Spacecraft Communication Systems; Spacecraft Flight Testing.

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Signal-Strength Dependence on Pointing

One approach to the on-orbit evaluation of antenna-boresight parameters is to slew the antenna relative to the direction to a ground-based transmitter with known radiated power characteristics. The pointing direction of the nominal antenna boresight with respect to the transmitter direction is determined through the orbital ephemeris and the attitude sensor(s). Since the received signal strength is functionally dependent on the antenna pointing, its measurement over various pointing directions yields a basis for boresight estimation.

An analytic description of the dependence of signal strength on pointing follows. Define the coordinate axes u, v, and w as orthogonal axes fixed in the antenna structure such that w is the nominal (a priori guess) of the antenna boresight, and u and v are the nominal cross-axes directions of the antenna. Let $\bar{l}$ be the unit vector along the direction from the spacecraft to the ground-based transmitter and denote the u, v, and w components of $\bar{l}$ by l_u, l_v, and l_w, respectively. Similarly, let $\bar{\mathbf{b}}$ be the unit vector along the (unknown) antenna boresight direction and denote its u, v, and w components by b_u, b_v, and b_w, respectively. If the angular departure of $\bar{l}$ from the boresight $\bar{\mathbf{b}}$ is small (less than the half-power beam width), then the received signal strength S in decibels is related to the pointing direction according to the matrix equation

$$S = S_0 - 12(\bar{\lambda} - \bar{\beta})^T\Gamma^T\Delta\Gamma(\bar{\lambda} - \bar{\beta}) \tag{1}$$

where S_o is the link signal strength corresponding to alignment of $\bar{l}$ and $\bar{\mathbf{b}}$; Γ describes a rotation through an angle γ about the boresight axis $\bar{b}$, i.e.,

$$\Gamma = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \tag{2}$$

The value Δ specifies the two orthogonal half-power beam widths δ_u and δ_v according to

$$\Delta = \begin{bmatrix} 1/\delta_u^2 & 0 \\ 0 & 1/\delta_u^2 \end{bmatrix} \tag{3}$$

and $\bar{\lambda}$ and $\bar{\beta}$ are the two-dimensional vector projections of $\bar{l}$ and $\bar{b}$ given by

$$\bar{\lambda} = \begin{pmatrix} l_u \\ l_v \end{pmatrix} \tag{4}$$

$$\bar{\beta} = \begin{pmatrix} b_u \\ b_v \end{pmatrix} \tag{5}$$

It is emphasized that the signal strength is a function of the two pointing variables l_u and l_v, which comprise the vector $\bar{\lambda}$, and of the six antenna parameters b_u, b_v, δ_u, δ_v, γ, and S_o. Thus, one writes that $S = S(l_u, l_v, b_u, b_v, \delta_u, \delta_v, \gamma, S_o)$ or that

$S = S(\bar{\lambda}, \bar{a})$, using the definition of $\bar{\lambda}$ and an antenna parameter vector according to

$$\bar{a} = [a_1 a_2 a_3 a_4 a_5 a_6]^T = [b_u b_v \delta_u \delta_v \gamma S_o]^T \tag{6}$$

A graphical representation of signal strength is shown in Fig. 1. The ellipses shown are contours of constant signal strength. The maximum signal strength is S_o, and the (u,v) coordinates where $S = S_o$ are (b_u, b_v), the antenna boresight departure from the nominal boresight. The major and minor axes of the $S_o - 3$ db contour are the two beam widths δ_u and δ_v; the rotation of the axes is specified by γ.

Antenna Parameter Estimation

The problem is to estimate the antenna parameters b_u, b_v, δ_u, δ_v, γ, and S_o from measurements of signal strength. In theory, error-free measurements corresponding to six distinct pointing values are sufficient to determine the antenna parameters (signal strength and pointing angle measurements are substituted in Eq. (1), and the resulting set of six equations is solved). In practice, the measurements are subject to error, so the use of statistical estimation methods using redundant (more than six) measurements is warranted. It is reasonable to make assumptions concerning only the statistical properties of the measurement errors; the statistical estimation method adopted in this study is the weighted least squares estimation process.[1]

Suppose that the measurements of signal strength are sampled at various pointing directions; denote the measurements by $m_1, m_2, \ldots m_n$, corresponding to the pointing directions $\bar{\lambda}^1, \bar{\lambda}^2, \ldots \bar{\lambda}^n$. The residual q_i is defined as the difference between the measured signal strength m_i and the predicted signal strength, which is based on an assumed set of antenna parameters. The analytic expression is

$$q_i = m_i - S(\bar{\lambda}^i, \bar{\alpha}) \tag{7}$$

where S is the scalar valued function defined in Eq. (1) and $\bar{\alpha}$ is an assumed antenna parameter vector identified by

$$\bar{\alpha} = [\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6]^T = [\tilde{b}_u \tilde{b}_v \tilde{\delta}_u \tilde{\delta}_v \gamma \tilde{S}_o]^T$$

where $\tilde{b}_u$ denotes the assumed value of b_u, etc. The net loss (risk or average loss for a fixed data set) is defined as the sum of the square of residuals weighed by the covariance matrix of measurement errors. Assume that the measurement errors are expected to be zero and that the covariance matrix of the measurement errors, called W, is known. The net loss Q is defined as

$$Q = [\bar{m} - \bar{\mu}(\bar{\alpha})]^T W^{-1} [\bar{m} - \bar{\mu}(\bar{\alpha})] \tag{8}$$

where $\bar{m}$ is the n vector with components $m_1, m_2, \ldots m_n$, and $\bar{\mu}$ is the n vector with components $S(\bar{\lambda}^1,\bar{\alpha})$, $S(\bar{\lambda}^2,\bar{\alpha})$, $\ldots$ $S(\bar{\lambda}^n,\bar{\alpha})$.

The best estimate of the antenna parameters is that vector value of α, called $\hat{\alpha}$, for which Q is a minimum. To express this analytically, the best estimate $\hat{\alpha}$ is the solution to the set of equations $\partial Q/\partial \alpha_i|_{\bar{\alpha}=\hat{\alpha}} = 0$, $i = 1,2, \ldots 6$, or the solution to the vector equation

$$J^T(\hat{\alpha}) \cdot W^{-1}[\bar{m} - \bar{\mu}(\hat{\alpha})] = 0 \tag{9}$$

where $J(\bar{\alpha})$ denotes the Jacobian matrix of partial derivatives with element $J_{ij}(\bar{\alpha}) = \partial \mu_i/\partial \alpha_j|_{\bar{\alpha}=\tilde{\alpha}} = \partial S(\lambda^i,\alpha)/\partial \alpha_j|_{\bar{\alpha}=\tilde{\alpha}}$. Expressions for these partial derivatives are listed in the Appendix. The resulting set of equations is nonlinear; its suggested solution is the evaluation of a sequence of linear equations. Specifically, approximate $\bar{\mu}$ by the two-term Taylor series expansion about a nominal value $\hat{\alpha}^k$, i.e., according to

$$\bar{\mu} = \bar{\mu}(\hat{\alpha}^k) + J(\hat{\alpha}^k)(\hat{\alpha} - \hat{\alpha}^k) \tag{10}$$

and approximate $J^T(\hat{\alpha})$ in Eq. (9) by $J^T(\hat{\alpha}^k)$. Using these approximations results in a recursive relationship.

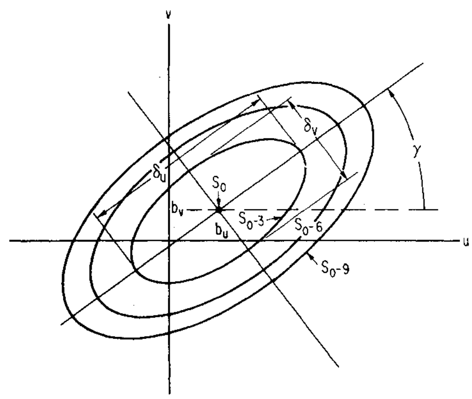

Fig. 1 Signal strength and antenna parameters.

$$\hat{\alpha}^{k+1} = \hat{\alpha}^k + [J^T(\hat{\alpha}^k) W^{-1} J(\hat{\alpha}^k)]^{-1} J^T(\hat{\alpha}^k) W^{-1} \times [\bar{m} - \bar{\mu}(\hat{\alpha}^k)] \tag{11}$$

It is argued, without proof, that if the rank of J is six, the sequence $\hat{\alpha}^0, \hat{\alpha}^1, \hat{\alpha}^2, \ldots$ converges to $\hat{\alpha}$, the (nonlinear) estimate of the antenna parameters that is best in the sense of minimizing Q.

Estimation Error

Use of the formulas yields an estimate $\hat{\alpha}$ of the antenna parameter vector. The error in this estimate is defined as $\bar{a} - \hat{\alpha}$, $\bar{a}$ being the true antenna parameter vector. Since the true value $\bar{a}$ is unknown, discussion is restricted to the statistical properties of the estimation error. Moreover, since closed expressions for $\hat{\alpha}$ are not available, only approximations based on the linearization of $\bar{\mu}$ about $\hat{\alpha}$ are apparent. Specifically, where E denotes expectation, the mean of estimation errors $E(\bar{a} - \hat{\alpha})$ is approximately zero (under the assumption of zero mean measurement errors) and the covariance matrix of estimation errors is approximated by

$$[J^T(\hat{\alpha}) W^{-1} J(\hat{\alpha})]^{-1}$$

Computer Simulation

Algorithms have been developed for a least squares estimation of antenna boresight parameters. To demonstrate the efficacy of these algorithms, computer simulation results of two test cases are presented. Consider an antenna with the nominal values: $b_u = b_v = 0$, $\delta_u = \delta_v = 0.02$ rad, $S_o = 0$ db, and $\gamma = 0$. Suppose that a raster scan of the antenna is performed such that signal-strength measurements are collected at the forty-five pointing directions indicated by dots in Fig. 2. Further suppose that e_i, the errors in the mea-

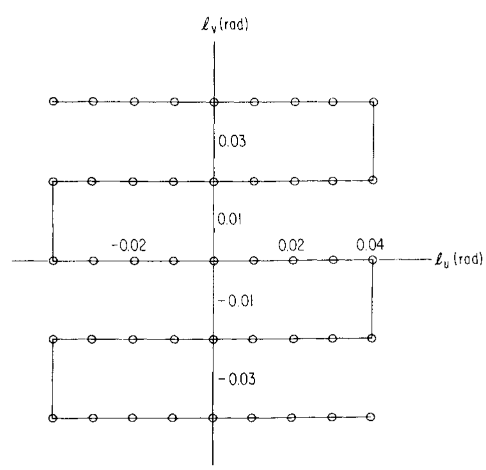

Fig. 2 Raster scan.

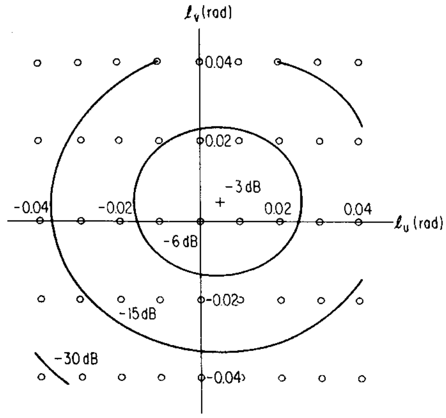

Fig. 3 Signal strength contours: Case 1.

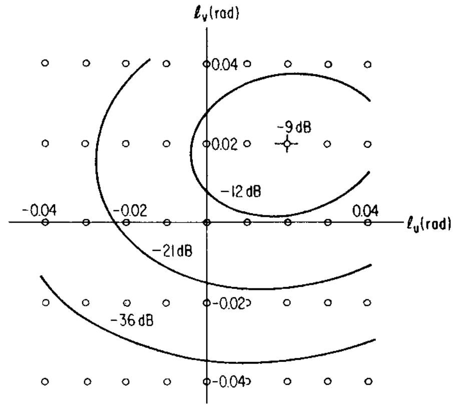

Fig. 4 Signal strength contours: Case 2.

surements, are additive and random of normal probability distribution with zero mean and 3-db standard deviation, so that

$$m_i = S(\bar{\lambda}^i, \bar{a}) + e_i$$

and

$$E(m_i) = S(\bar{\lambda}^i, \bar{a}), \ [E(e_i^2)]^{1/2} = 3 \text{ db}$$

where $S(\bar{\lambda}^i, \bar{a})$ is the error-free signal strength specified by the true antenna parameter vector $\bar{a}$.

A computer simulation has been formulated using the previous expression for the measurements and the pointing directions indicated in the raster scan. The true antenna parameter is specified in the simulation and knowledge of its value allows evaluation of estimates $\hat{a}$, i.e., the error $\bar{a} - \hat{a}$. Two test cases are considered: In the first case, the deviation from nominal values is small; $\bar{a}$ = (0.005, 0.005, 0.021, 0.019, 0.125, −3). In the second case, the deviation from nominal values is large; $\bar{a}$ = (0.020, 0.020, 0.024, 0.018, 0.1250, −9). The antenna gain patterns corresponding to these values of $\bar{a}$ are shown in Figs. 3 and 4, relative to the raster pattern.

The expression for the covariance matrix of estimation errors $[J^T(\hat{a})W^{-1}J(\hat{a})]^{-1}$, evaluated under the assumption that $\hat{a} = \bar{a}$, is listed below for each of the two cases (the physical units are rad², db², and rad-db).

Case 1: symmetric

7.66×10^{-7}	5.94×10^{-8}	6.77×10^{-7}	-1.24×10^{-8}	8.63×10^{-6}	-1.18×10^{-4}
	4.43×10^{-7}	4.66×10^{-8}	2.92×10^{-7}	5.74×10^{-5}	-1.07×10^{-4}
		1.35×10^{-6}	1.70×10^{-8}	-4.68×10^{-5}	-5.00×10^{-4}
			5.71×10^{-7}	2.24×10^{-5}	-3.61×10^{-4}
				4.37×10^{-2}	1.29×10^{-2}
					6.61×10^{-1}

Case 2: symmetric

7.98×10^{-6}	1.63×10^{-6}	4.09×10^{-6}	-3.07×10^{-8}	-4.40×10^{-5}	2.86×10^{-3}
	3.21×10^{-6}	8.20×10^{-7}	7.37×10^{-7}	8.90×10^{-5}	1.86×10^{-3}
		2.33×10^{-6}	6.81×10^{-8}	4.03×10^{-5}	1.18×10^{-3}
			3.24×10^{-7}	1.02×10^{-5}	1.46×10^{-4}
				6.53×10^{-3}	4.32×10^{-2}
					2.28

A sample of eighteen separate computer runs was considered, using random number generation of the e_i value. The sample means of the estimation errors are listed below in units of rad and db:

Case 1: (−0.000031, 0.000445, −0.000347, 0.000376, 0.0427, −0.462)
Case 2: (−0.000531, 0.000714, −0.000407, 0.000257, 0.0219, −0.0727)

The sample covariance matrix of estimation errors agrees with that found by evaluating $(J^TWJ)^{-1}|_{\hat{a}=\bar{a}}$.

Numerical stability of the iteration for nonlinear estimates was observed in every computer run; i.e., the magnitude of the difference of successive estimates approaches zero as iterations are performed. The importance of the iteration for nonlinear estimates is observed especially in the second test case, where the deviation of true parameters from the nominal values is large. The estimate resulting from the first iteration is a linear estimate; a typical sequence of estimates is given below:

$$\begin{aligned}
\hat{a}_0 &= (\ 0\ ,\ 0\ ,\ 0.02\ ,\ 0.02\ ,\ 0\ ,\ 0\ ) \\
\hat{a}_1 &= (0.0120, 0.0222, 0.0230, 0.0182, \ 0\ , -29.02) \\
\hat{a}_2 &= (0.0209, 0.0201, 0.0242, 0.0184, 0.2874, -9.43) \\
\hat{a}_3 &= (0.0215, 0.0217, 0.0248, 0.0183, 0.2515, -7.12) \\
\hat{a}_4 &= (0.0220, 0.0216, 0.0249, 0.0183, 0.2562, -6.97) \\
\hat{a}_5 &= (0.0220, 0.0216, 0.0249, 0.0183, 0.2562, -6.96) \\
\hat{a}_6 &= (0.0220, 0.0216\ 0.0249, 0.0183, 0.2562, -6.96) \\
\bar{a} &= (0.0200, 0.0200, 0.0240, 0.0180, 0.2500, -9.00)
\end{aligned}$$

The nonlinear estimate error is $\bar{a} - \hat{a}_6$ = (−0.0020, −0.0016, −0.0009, −0.0003, −0.0062, −2.04). This is a noticeable improvement on the linear estimate error $\bar{a} - \hat{a}_1$ = (0.0080, −0.0022, 0.0010, −0.0002, 0.2500, 20.02).

Auxiliary Formulas for Antenna Pointing

The previous analysis dealt primarily with the algorithms for antenna parameter estimation from signal strength measurements and from the pointing directions. The pointing directions λ^i are dependent on the spacecraft (attitude reference axes) orientation, the antenna orientation relative to the attitude reference orientation, and the spacecraft position relative to the ground-based transmitter location. The equations for antenna pointing are listed below. The ground transmitter position vector $\bar{T}$ expressed in the inertial coordinates is defined as

$$\bar{T}^I = r_T \begin{pmatrix} \cos\lambda_T \sin\eta \\ \sin\lambda_T \\ \cos\lambda_T \cos\eta \end{pmatrix}$$

where r_T is the ground transmitter radial distance and

$$\eta = l_T - l_\Omega + \omega_E(t - \tau_\Omega)$$

The spacecraft position vector $\bar{S}$ expressed in orbital coordinates is defined as

$$\bar{S}^0 = a(1 - e\cos E) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

and the eccentric anomaly is related to time by the relation

$$(t - \tau_\pi) = (E - e\sin E)(a^3/K)^{1/2}$$

Note that the two epochs τ_π and τ_Ω are related through the orbital parameters. Particularly, note that

$$(1 + e)^{1/2} \tan\tfrac{1}{2}E' = (1 - e)^{1/2} \tan\tfrac{1}{2}\omega$$

$$\tau_\pi - \tau_\Omega = (E' - e\sin E')(a^3/K)^{1/2}$$

The vector difference $\bar{T} - \bar{S}$ defines the spacecraft-to-transmitter position. Therefore the unit vector $\bar{l}$, defining the pointing direction, is $\bar{T} - \bar{S}$ normalized to unit length, i.e., $\bar{l} = (\bar{T} - \bar{S})/|\bar{T} - \bar{S}|$. If R_{0I} denotes the transformation from inertial coordinates to orbital coordinates, $\bar{l}$ is expressed in orbital coordinates as

$$\bar{l}^0 = (R_{0I}\bar{T}^I - \bar{S}^0)/|R_{0I}\bar{T}^I - \bar{S}^0|$$

For the inertial and orbital coordinates defined in the Nomenclature:

$$R_{0I} = \begin{bmatrix} \cos(\omega + \nu)\cos i & \cos(\omega + \nu)\sin i & -\sin(\omega + \nu) \\ \sin i & -\cos i & 0 \\ -\sin(\omega + \nu)\cos i & -\sin(\omega + \nu)\sin i & -\cos(\omega + \nu) \end{bmatrix}$$

If R_{S0} denotes the transformation from orbital- to spacecraft-attitude reference coordinates, the pointing direction $\bar{l}$ is expressed in the spacecraft coordinates by $\bar{l}^S = R_{S0}\bar{l}^0$.

For the defined Euler angle sequence one finds that

$$R_{S0} = \begin{bmatrix} C\theta C\psi & C\theta S\psi & -S\theta \\ S\theta S\Phi C\psi - C\Phi S\psi & S\theta S\Phi S\psi + C\Phi C\psi & C\theta S\Phi \\ S\theta C\Phi C\psi + S\Phi S\psi & S\theta C\Phi S\psi - S\Phi C\psi & C\theta C\Phi \end{bmatrix}$$

where $C\theta$, $C\psi$, $S\psi$, $S\theta$, and $S\Phi$ denote $\cos\theta$, $\cos\psi$, $\sin\psi$, $\sin\theta$, and $\sin\Phi$. Whenever ψ, Φ, and θ are small angles, use of the approximation

$$R_{S0} \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \Phi \\ \theta & -\Phi & 1 \end{bmatrix}$$

is appropriate to simplify calculations. If R_{AS} denotes the transformation from the spacecraft attitude reference coordinates to the antenna coordinates, the desired expression for the pointing direction in the antenna coordinates is $\bar{l} = R_{AS}\bar{l}^S$.

Specification of the transformation matrix R_{AS} depends on the spacecraft configuration. One general specification of R_{AS} is developed in the following discussion.

Consider a set of orthogonal coordinates u', v', w', fixed in the spacecraft. If the antenna is not gimballed, identify u', v', w' with u, v, w (the nominal antenna coordinates); if the antenna is gimballed through the usual two orthogonal gimbal rotations, identify u' with the outer and v' with the inner gimbal axis. Now R_{AS} decomposes into the product of two transformation matrices

$$R_{AS} = R_{AA'}R_{A'S}$$

where $R_{A'S}$ describes the rotation from the spacecraft coordinates to the u', v', w' coordinates, and $R_{AA'}$ describes the gimbal angle rotations. If the antenna is not gimballed, $R_{AA'}$ is the identity matrix; if the antenna is gimballed

$$R_{AA'} = \begin{bmatrix} \cos\xi_2 & 0 & \sin\xi_2 \\ 0 & 1 & 0 \\ \sin\xi_2 & 0 & \cos\xi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi_1 & \sin\xi_1 \\ 0 & -\sin\xi_1 & \cos\xi_1 \end{bmatrix}$$

where ξ_1 and ξ_2 are the outer and inner gimbal angles. Specification of the spacecraft-antenna configuration is now placed in the constant matrix $R_{A'S}$. This configuration is specified in some ordered rotation sequence by Euler angles or by an equivalent axis rotation. In the latter case, one specifies a single axis fixed in the X, Y, Z coordinates and a rotation angle about that axis to the new u', v', w' coordinates. If the direction cosines of the axis are l_0, m_0, and n_0 with respect to the X, Y, and Z axes, and if ξ denotes the rotation angle, then

$$R_{A'S} = \begin{bmatrix} (1 - l_0^2)\cos\xi + l_0^2 & l_0 m_0 (1 - \cos\xi) & -m_0 \sin\xi \\ l_0 m_0 (1 - \cos\xi) & (1 - m_0^2)\cos\xi + m_0^2 & -l_0 \sin\xi \\ m_0 \sin\xi & l_0 \sin\xi & \cos\xi \end{bmatrix}$$

Conclusions

This paper has developed algorithms for estimating the antenna parameters; in particular, for the antenna boresight orientation relative to the spacecraft attitude sensor coordinates. The resulting estimate is based upon a standard antenna gain pattern model modified for an asymmetrical beam and is a best estimate in the weighted least squares sense. The data used for the parameter estimation are principally measurements of received signal strength, and the use of the statistical estimation using redundant data seems particularly appropriate as significant random noise components are inherent to the measurements.

Computer simulation of the estimation process was performed and sample results obtained are in agreement with the theoretical statements. Numerical stability for the nonlinear estimate was observed in every case. The boresight orientation estimate error was found to have a sample mean of $\frac{1}{100}$ beamwidth and a sample standard deviation of $\frac{1}{10}$ beam-width. The sample covariance matrix was seen to agree with the theoretical values. A significant portion of the data reduction involves the transformation of orbital parameters and attitude sensor data to the determination of the antenna-pointing direction. The relevant formulas are listed in the previous section.

Appendix

Expressions for the elements of the Jacobian matrix J_{ij} are listed here.

$$J_{i1} = \partial S_i/\partial b_u = 24(\Sigma_{11}\omega_x + \Sigma_{12}\omega_y)$$

$$J_{i2} = \partial S_i/\partial b_v = 24(\Sigma_{12}\omega_x + \Sigma_{22}\omega_y)$$

$$J_{i3} = \partial S_i/\partial \delta_u = 24(\omega_x \cos\gamma + \omega_y \sin\gamma)^2/\delta_u^3$$

$$J_{i4} = \partial S_i/\partial \delta_v = 24(\omega_x \sin\gamma - \omega_y \cos\gamma)^2/\delta_v^3$$

$$J_{i6} = \partial S_i/\partial S_0 = 1$$

$$J_{i5} = \partial S_i/\partial\gamma = 24(\delta_u^{-2} - \delta_v^{-2})[(\omega_x^2 - \omega_y^2)\sin\gamma\cos\gamma - \omega_x\omega_y(\cos^2\gamma - \sin^2\gamma)]$$

where

$$\omega_x = l_u^i - b_u, \quad \omega_y = l_u^i - b_v,$$

$$\Sigma_{11} = \delta_u^{-2}\cos^2\gamma + \delta_v^{-2}\sin^2\gamma$$

$$\Sigma_{12} = (\delta_u^{-2} - \delta_v^{-2})\sin\gamma\cos\gamma,$$

$$\Sigma_{22} = \delta_u^{-2}\sin^2\gamma + \delta_v^{-2}\cos^2\gamma$$

Reference

[1] Deutsch, R., "Estimation Theory", Prentice Hall, Englewood Cliffs, N.J., 1965, pp. 72–77.